Mean field linear–quadratic control: Uniform stabilization and social optimality

Bing-Chang Wang\textsuperscript{a,}\textsuperscript{*}, Huanshui Zhang\textsuperscript{b,a}, Ji-Feng Zhang\textsuperscript{c,d}

\textsuperscript{a} School of Control Science and Engineering, Shandong University, Jinan 250061, China
\textsuperscript{b} College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266590, China
\textsuperscript{c} Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
\textsuperscript{d} School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100149, China

\textbf{A B S T R A C T}

This paper is concerned with uniform stabilization and social optimality for general mean field linear–quadratic control systems, where subsystems are coupled via individual dynamics and costs, and the state weight is not assumed with the definiteness condition. For the finite-horizon problem, we first obtain a set of forward–backward stochastic differential equations (FBSDEs) from variational analysis, and construct a feedback-type control by decoupling the FBSDEs. For the infinite-horizon problem, by using solutions to two Riccati equations, we design a set of decentralized control laws, which is further proved to be asymptotically social optimal. Some equivalent conditions are given for uniform stabilization of the systems in different cases, respectively. Finally, the proposed decentralized controls are compared to the asymptotic optimal strategies in previous works.

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1. Introduction

Mean field games have drawn increasing attention in many fields including system control, applied mathematics and economics (Bensoussan, Frehse, & Yam, 2013; Caines, Huang, & Malhamé, 2017; Gomes & Saude, 2014). The mean field game involves a very large population of small interacting players with the feature that while the influence of each one is negligible, the impact of the overall population is significant. By combining mean field approximations and individual’s best response, the dimensionality difficulty is overcome. Mean field games and control have found wide applications, including smart grids (Chen, Busic, Busic, & Meyn, 2017; Li, Ma, Li, Chen, & Gu, 2019; Ma, Callaway, & Hiskens, 2013), finance, economics (Chan & Sircar, 2015; Guéant, Lasry, & Lions, 2011; Wang & Huang, 2019), and social sciences (Bauso, Tentabile, & Basar, 2016), etc.

By now, mean field games have been intensively studied in the LQ (linear–quadratic) framework (Bensoussan, Sung, Yam, & Yung, 2016; Elliott, Li, & Ni, 2013; Huang, Caines, & Malhamé, 2007; Li & Zhang, 2008; Moon & Basar, 2017; Wang & Zhang, 2012b). Huang et al. developed the Nash certainty equivalence (NCE) based on the fixed-point method and designed an $\epsilon$-Nash equilibrium for mean field LQ games with discount costs by the NCE approach (Huang et al., 2007). The NCE approach was then applied to the cases with long run average costs (Li & Zhang, 2008) and with Markov jump parameters (Wang & Zhang, 2012b), respectively. The works (Bensoussan et al., 2016; Carmona & Delarue, 2013) employed the adjoint equation approach and the fixed-point theorem to obtain sufficient conditions for the existence of the equilibrium strategy over a finite horizon. For other aspects of mean field games, readers are referred to Carmona and Delarue (2013), Huang, Malhamé, and Caines (2005), Lasry and Lions (2007) and Yin, Mehta, Meyn, and Shanbhag (2012) for nonlinear mean field games, Weintraub, Benard, and Van Roy (2008) for oblivious equilibrium in dynamic games, Huang (2010) and Wang and Zhang (2012a) for mean field games with major players, Huang and Huang (2017) and Moon and Basar (2017) for robust mean field games.

Besides noncooperative games, social optima in mean field models have also attracted much interest. The social optimum control refers to that all the players cooperate to optimize the common social cost—the sum of individual costs, which is a type of team decision problem (Ho, 1980). Huang et al. considered social optima in mean field LQ control, and provided an asymptotic
team-optimal solution (Huang, Caines, & Malhamé, 2012). Wang and Zhang (2017) investigated the mean field social optimal problem where the Markov jump parameter appears as a common source of randomness. For further literature, see Huang and Nguyen (2016) for social optimal in mixed games, Aranbegydi and Mahajan (2015) for team-optimal control with finite population and partial information.

Most previous results on mean field games and control were given by using the fixed-point method (Cardaliaguet, 2012; Carmona & Delarue, 2013; Huang et al., 2007, 2012; Li & Zhang, 2008; Wang & Zhang, 2012a, 2017). However, the fixed-point analysis (e.g., from the contraction mapping theorem) is sometimes conservative, particularly for high-dimensional systems. In this paper, we solve the problem by decoupling directly high-dimensional forward–backward stochastic differential equations (FBSDEs). In recent years, some progress has been made for study of the optimal LQ control by tackling the FBSDEs. See Sun, Li, and Yong (2016), Zhang, Qi, and Fu (2019), Zhang and Xu (2017) and Yong (2013) for details.

This paper investigates uniform stabilization and social optimality for linear–quadratic mean field control systems, where subsystems (agents) are coupled via dynamics and individual costs. The state weight Q is not limited to positive semi-definite. This model can be taken as a generation of robust mean field control problems (Huang & Huang, 2017; Moon & Basar, 2017; Wang & Huang, 2017). However, the fixed-point conditions are given for uniform stabilization in different cases. In Section 5, some numerical examples are given to show the effectiveness of the proposed control laws. Section 6 concludes the paper.

The following notation will be used throughout this paper.

\[ \| \cdot \| \text{ denotes the Euclidean vector norm or Frobenius matrix norm.} \]

For a vector \( x \) and a matrix \( Q \), \( \| x \|^2_Q = x^TQx \), \( \| \cdot \|_Q \) is the trace of the matrix \( Q \), and \( Q > 0 \) (\( Q \geq 0 \)) means that \( Q \) is positive definite (positive semidefinite). For two vectors \( x, y \), \( (x, y) = x^Ty \). \( C([0, T], \mathbb{R}^n) \) is the space of all \( \mathbb{R}^n \)-valued continuous functions defined on \( [0, T] \), and \( C_{\rho/2}([0, \infty), \mathbb{R}^n) \) is a subspace of \( C([0, \infty), \mathbb{R}^n) \) which is given by \( \{ f \mid \int_0^\infty e^{-\rho t} \| f(t) \|^2 dt < \infty \} \). \( L_2^0(0, T; \mathbb{R}^n) \) is the space of all \( F \)-adapted \( \mathbb{R}^n \)-valued processes \( x(\cdot) \) such that \( \mathbb{E} \int_0^T \| x(t) \|^2 dt < \infty \). For convenience of presentation, we use \( C, C_1, C_2, \ldots \) to denote generic positive constants, which may vary from place to place.

### 2. Problem description

Consider a large population system with \( N \) agents. Agent \( i \) evolves by the following stochastic differential equation:

\[
\begin{align*}
\dot{x}_i(t) & = [Ax_i(t) + Bu_i(t) + Gx_i(t)] + f(t) dt + \sigma(t) dW_i(t), \quad 1 \leq i \leq N, \\
& \text{where } x_i \in \mathbb{R}^n \text{ and } u_i \in \mathbb{R}^n \text{ are the state and input of the ith agent.}
\end{align*}
\]

For the finite-horizon social control problem, we first obtain a set of FBSDEs by examining the variation of the social cost, and give centralized feedback-type control laws by decoupling the FBSDEs. With mean field approximations, we design a set of decentralized control laws. By exploiting the uniform convexity property of the problem, the decentralized controls are further shown to have asymptotic social optimality. For the infinite-horizon case, we design a set of decentralized control laws by using solutions of two Riccati equations, which is shown to be asymptotically social optimal. Some equivalent conditions are further given for uniform stabilization of all the subsystems when the state weight \( Q \) is positive semi-definite or only symmetric. Furthermore, the explicit expressions of optimal social costs are given in terms of the solutions to two Riccati equations, and the proposed decentralized control laws are compared to the feedback strategies in previous works. Finally, some numerical examples are given to illustrate the effectiveness of the proposed control laws.

The main contributions of the paper are summarized as follows.

- We first obtain necessary and sufficient existence conditions of finite-horizon centralized optimal control by variational analysis, and then design a feedback-type decentralized control by tackling FBSDEs with mean field approximations.
- In the case \( Q \geq 0 \), the necessary and sufficient conditions are given for uniform stabilization of the systems with the help of the system’s observability and detectability.
- In the case that \( Q \) is indefinite, the necessary and sufficient conditions are given for uniform stabilization of the systems using the Hamiltonian matrices.
- The asymptotically optimal decentralized controls are obtained under very basic assumptions (without verifying the fixed-point condition). The corresponding social costs are explicitly given by virtue of the solutions to two Riccati equations.

The organization of the paper is as follows. In Section 2, the socially optimal control problem is formulated. In Section 3, we construct asymptotically optimal decentralized control laws by tackling FBSDEs for the finite-horizon case. In Section 4, for the infinite-horizon case, the asymptotically optimal controls are designed and analyzed, and some equivalent conditions are further given for uniform stabilization in different cases. In Section 5, some numerical examples are given to show the effectiveness of the proposed control laws. Section 6 concludes the paper.

(P): Seek a set of decentralized control laws to optimize social cost for the system (1)–(2), i.e., \( \inf_{u_i \in U_i} J_{soc} \), where \( J_{soc} = \sum_{i=1}^N J_i(u) \).

**Remark 2.1.** The related results can be extended to the case of multidimensional Brownian motions trivially. Here we consider that \( \sigma(t) \) is time-varying and satisfies some growth rate. For
convenience of the statement, we assume $W_i$ is scalar and $\sigma \in C_{r/2}([0, \infty), \mathbb{R}^n)$. For the finite-horizon problem, our results still hold for the case that the matrices $A, B, G, \ldots$ depend on $t$.

Assume

(A1) The initial states of agents $x_i(0), i = 1, \ldots, N$ are mutually independent and have the same mathematical expectation. $x_i(0) = x_0, \mathbb{E}x_i(0) = \bar{x}_i, i = 1, \ldots, N$. There exists a constant $C_0$ (independent of $N$) such that $\max_{1 \leq t \leq N} \mathbb{E}\|x_i(0)\|^2 \leq C_0$.

3. The finite-horizon problem

For the convenience of design, we first consider the following finite-horizon problem.

(P1) \[
\inf_{u \in L_2^2(0, T; \mathbb{R}^n)} J_{\text{sec}}^f(u),
\]

where $J_{\text{sec}}^f(u) = \sum_{i=1}^N J_i^f(u)$ and $F_t = \sigma \left( \bigcup_{j=1}^N F_j^f \right)$. Here

\[
J_i^f(u) = \mathbb{E} \int_0^T e^{-\rho t} \left\{ \|y_i(t) - \Gamma y_i(N)(t)\|^2_Q + \|\mu_i(t)\|^2_R \right\} dt.
\]

We first give equivalent conditions for the convexity of (P1).

**Proposition 3.1.** (i) Problem (P1) is convex in $u$ if and only if for any $u_i \in L_2^2(0, T; \mathbb{R}^n), i = 1, \ldots, N$,

\[
\sum_{i=1}^N \int_0^T e^{-\rho t} \left[ \|y_i(t) - \Gamma y_i(N)(t)\|^2_Q + \|\mu_i(t)\|^2_R \right] dt \geq 0,
\]

where $y_i(0) = \sum_{i=1}^N y_i/N$ and $y_i$ satisfies

\[
dy_i(t) = [A y_i(t) + G y_i(N)(t) + B u_i(t)] dt,
\]

$y_i(0) = 0, i = 1, 2, \ldots, N$.

(ii) Problem (P1) is uniformly convex in $u$ if and only if for any $u_i \in L_2^2(0, T; \mathbb{R}^n)$, there exists $\gamma > 0$ such that

\[
\sum_{i=1}^N \int_0^T e^{-\rho t} \left[ \|y_i(t) - \Gamma y_i(N)(t)\|^2_Q + \|\mu_i(t)\|^2_R \right] dt \geq \gamma \sum_{i=1}^N \int_0^T e^{-\rho t} \|\mu_i(t)\|^2_R dt.
\]

**Proof.** Let $x_i$ and $\bar{x}_i$ be the state processes of agent $i$ with the control $v$ and $\bar{v}$, respectively. Take any $\lambda_1 \in [0, 1]$ and let $\lambda_2 = 1 - \lambda_1 > 0$. Then

\[
\lambda_1 J_{\text{sec}}^f(v) + \lambda_2 J_{\text{sec}}^f(\bar{v}) = J_{\text{sec}}^f(\lambda_1 v + \lambda_2 \bar{v})
\]

\[= \lambda_1 \lambda_2 \sum_{i=1}^N \int_0^T \left[ \|x_i(t) - \bar{x}_i(t) - \Gamma(x_i(N) - \bar{x}_i(N))\|^2_Q + \|\mu_i(t) - \bar{\mu}_i(t)\|^2_R \right] dt.
\]

Denote $u = v - \bar{v}$, and $y_i = x_i - \bar{x}_i$. Thus, $y_i$ satisfies (4). By the definition of (uniform) convexity, the lemma follows. \qed

By examining the variation of $J_{\text{sec}}^f$, we obtain the necessary and sufficient conditions for the existence of centralized optimal control of (P1). To simplify the presentation later, we denote

\[
\mathcal{L} \triangleq \Gamma^T Q + Q \Gamma - \Gamma^T Q \Gamma,
\]

\[
\bar{\eta} \triangleq Q \eta - \Gamma^T Q \eta.
\]

**Theorem 3.1.** Suppose $R > 0$. Then (P1) has a set of optimal control laws if and only if Problem (P1) is convex in $u$ and the following equation system admits a set of solutions $(x_i, p_i, p_i^N, i = 1, \ldots, N)$:

\[
\begin{align*}
\dot{x}_i(t) & = (Ax_i(t) - BR^{-1}B^T p_i(t) + G x_i(N)(t) + f(t)) dt + \sigma(t) dW_i(t), \\
p_i(t) & = -\left[ (A - \rho I)^T p_i^N(t) + G^T p_i^N(t) + Q x_i(t) \right] dt + \mathcal{L} x_i(N)(t) + \bar{\eta}(t) dt + \sum_{j=1}^N p_i^N(t) dW_j(t), \\
x_i(0) & = x_0, \quad p_i(T) = 0, \quad i = 1, \ldots, N,
\end{align*}
\]

where $p_i^N(t) = \frac{1}{N} \sum_{j=1}^N p_i(t)$, and furthermore the optimal control is given by $\bar{u}_i(t) = -R^{-1}B^T p_i(t)$.

**Proof.** Suppose that $\bar{u}_i = -R^{-1}B^T p_i$, where $(p_i, p_i^N, i = 1, \ldots, N)$ is a set of solutions to the second equation in (5). Denote by $\bar{x}_i$ the state of agent $i$ under the control $\bar{u}_i$. For any $u_i \in L_2^2(0, T; \mathbb{R}^n)$ and $\theta \in \mathbb{R}$ ($\theta \neq 0$), let $u_i^\theta = u_i + \theta \bar{u}_i$. Denote $x_i^\theta$ the solution of the following perturbed state equation

\[
dx_i^\theta(t) = \left[ Ax_i^\theta(t) + B(u_i^\theta(t) + \theta \bar{u}_i(t)) + f(t) \\
+ \frac{G}{N} \sum_{i=1}^N x_i^\theta(t) \right] dt + \sigma(t) dW_i(t),
\]

$x_i^\theta(0) = x_0, i = 1, 2, \ldots, N$.

Let $y_i = (x_i^\theta - \bar{x}_i)/\theta$. It can be verified that $y_i$ satisfies (4). Then by Itô’s formula, for any $i = 1, \ldots, N$,

\[
0 = \mathbb{E}\left[ e^{-\rho T} p_i(T, y_i(T)) - (p_i(0), y_i(0)) \right] = \mathbb{E} \int_0^T e^{-\rho t} \left[ -(A - \rho I)^T p_i(t) + G^T p_i^N(t) + Q x_i(t) \right] dt
\]

\[+ \mathcal{L} x_i(N)(t) + \bar{\eta}(t), y_i(t)) dt + (G^T p_i^N(t), y_i(t)) dt,
\]

which implies

\[
0 = \sum_{i=1}^N \mathbb{E} \int_0^T e^{-\rho t} \left[ -(A - \rho I)^T p_i(t) + G^T p_i^N(t) + Q x_i(t) \right] dt
\]

\[+ \mathcal{L} x_i(N)(t) + \bar{\eta}(t), y_i(t)) dt + (G^T p_i^N(t), y_i(t)) dt.
\]

From (3), we have

\[
J_{\text{sec}}^f(\bar{u} + \theta \bar{u}) - J_{\text{sec}}^f(\bar{u}) = 2\theta I_1 + \theta^2 I_2
\]

where $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_N)$, and

\[
I_1 \triangleq \sum_{i=1}^N \int_0^T e^{-\rho t} \left[ (Q (x_i(t) - (\Gamma x_i(N) + \eta_i), y_i^\theta(t) - \Gamma y_i(N)(t) + (G^T p_i^N(t), (R_i(\bar{u}_i(t), u_i(t)))) dt,
\]

\[I_2 \triangleq \sum_{i=1}^N \int_0^T e^{-\rho t} \left[ \|y_i(t) - \Gamma y_i(N)(t)\|^2_Q + \|u_i(t)\|^2_R \right] dt.
\]

Note that (suppressing the time $t$)

\[
\sum_{i=1}^N \mathbb{E} \int_0^T e^{-\rho t} (Q (x_i(t) - (\Gamma x_i(N) + \eta_i)), y_i^\theta(t) - \Gamma y_i(N)(t)) dt
\]

\[= \mathbb{E} \int_0^T e^{-\rho t} \left[ \sum_{i=1}^N (x_i(t) - (\Gamma x_i(N) + \eta_i)) \cdot \frac{1}{N} \sum_{i=1}^N y_i(t) \right] dt.
\]
\[
\sum_{i=1}^{N} E \int_0^T e^{-\mu t} (\Gamma^T Q) \sum_{i=1}^{N} (\dot{x}_i - (\Gamma \dot{x}_N + \eta)), y_i) dt
\]

\[
= \sum_{i=1}^{N} E \int_0^T e^{-\mu t} (\Gamma^T Q (I - \Gamma \dot{x}_N - \eta)), y_i) dt.
\]

From (6), one can obtain that
\[
I_1 = E \int_0^T e^{-\mu t} \left[ \frac{\dot{y}_i}{P \eta_i} \right] dt
\]
\[
+ \sum_{i=1}^{N} E \int_0^T e^{-\mu t} \left[ -(\Gamma^T p^N(t) + Q \eta_i(t))ight. \\
\left. + \mathbb{E} x(t) + \dot{\eta}(t) + (A - \rho I)^T p_i + G^T p^N(t), y_i) dt \right]
\]
\[
= \sum_{i=1}^{N} E \int_0^T e^{-\mu t} (\mathbb{E} u_i + B^T p_i, u_i) dt.
\]

From (7), \( \dot{u} \) is a minimizer to Problem (P1) if and only if \( \dot{I}_2 = 0 \) and \( I_1 = 0 \). By Proposition 3.1, \( \dot{I}_2 \geq 0 \) if and only if (P1) is convex. \( I_1 = 0 \) is equivalent to
\[
\dot{u}_i = -R^{-1} B^T p_i.
\]
Thus, we have the optimality system (5). This implies that (5) admits a solution \( (\hat{x}_i, \dot{p}_i, \dot{x}_j, i, j = 1, \ldots, N) \).

On the other hand, if the equation system (5) admits a solution \( (\hat{x}_i, \dot{p}_i, \dot{x}_j, i, j = 1, \ldots, N) \), then \( \dot{u}_i = -R^{-1} B^T p_i \). If (P1) is convex, then \( \dot{u} \) is a minimizer to Problem (P1).

It follows from (5) that
\[
\begin{align*}
\dot{x}^N(t) & = [(A + G) \dot{x}^N(t) - BR^{-1} B^T p^N(t) + f(t)] dt \\
+ & \frac{1}{N} \sum_{i=1}^{N} \sigma(t) dW_i(t),
\end{align*}
\]
\[
\begin{align*}
\dot{p}^N(t) & = -\left[ (A + G - \rho I)^T p^N(t) \\
& + (I - \Gamma)^T Q (I - \Gamma \dot{x}^N(t) - \dot{\eta}(t)) \right] dt \\
+ & \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \dot{p}_i(t) dW_j(t),
\end{align*}
\]
\[
\begin{align*}
\dot{x}^N(0) & = \frac{1}{N} \sum_{i=1}^{N} x_{i0}, \ p^N(T) = 0.
\end{align*}
\]

Let \( p_i(t) = P(t) x_i(t) + K(t) \dot{x}_N(t) + \dot{s}(t), t \geq 0 \). Then by (5), (9) and Itô’s formula (suppressing the time t),
\[
\begin{align*}
\dot{p}_i & = \dot{P}(t) x_i(t) + P(t) \dot{x}_i(t) + K(t) \dot{\dot{x}}_N(t) + \dot{s}(t) \\
& + G \dot{x}^N(t) + f(t) + \sigma dW_i(t) + (\dot{s} + \dot{K} x_N) dt \\
& = \left[ (A - \rho I)^T (P(t) x_i(t) + K(t) \dot{x}_N(t) + s(t)) + G^T (P(t) + K(t)) \dot{x}_N(t) + s(t) \right] dt \\
& + \frac{1}{N} \sum_{i=1}^{N} \sigma(t) dW_i(t)
\end{align*}
\]
This implies \( \dot{p}_i = \frac{1}{2} K \sigma + P \sigma, \dot{p}_j = \frac{1}{2} K \sigma, \ j \neq i, \)
\[
\rho P(t) = \dot{P}(t) + A^T P(t) + P(t) A + Q \\
- P(t) B R^{-1} B^T P(t), \ P(T) = 0.
\]
\[
\rho K(t) = \dot{K}(t) + (A + G)^T K(t) + K(t) (A + G) + G^T P(t) \\
+ P(t) G = (P(t) + K(t)) BR^{-1} B^T P(t) + K(t)) \\
+ P(t) B R^{-1} B^T P(t) - \zeta, \ K(T) = 0.
\]
\[
\rho s(t) = \dot{s}(t) + (-A^T B^T (P + K)^T s(t) \\
+ (P + K) f(t) - \dot{\eta}(t), \ s(T) = 0.
\]

\[\text{Remark 3.1.} \text{ Note that (11) is not a standard Riccati equation. Its solvability may be referred to Abou-Kandil, Freiling, Ionescu, and Jank (2003). In particular, by Theorem 4.3 in Ma and Yong (1999, Chapter 2), if det} \left\{ \begin{array}{c} \begin{array}{ccc} \Lambda & = A - \frac{\nu}{2} I - BR^{-1} B^T \Lambda \ , \\
-\nu I & -A^T + \frac{\nu}{2} I \end{array} \end{array} \right\} > 0 \text{ with} \]
\[
A = \left[ \begin{array}{ccc} A - \frac{\nu}{2} I & -BR^{-1} B^T \\
-\nu I & -A^T + \frac{\nu}{2} I \end{array} \right], \text{ then we have} \]
\[
P(t) = \left[ \begin{array}{ccc} \Lambda & 0 \\
0 & I \end{array} \right]^{-1} \left[ \begin{array}{ccc} \Lambda & 0 \\
0 & I \end{array} \right].
\]

\[\text{Remark 3.2.} \text{ Denote } \Pi = P + K. \text{ Then from (10) and (11), } \Pi \text{ satisfies} \]
\[
\rho \Pi(t) = \dot{\Pi}(t) + (A + G)^T \Pi(t) + \Pi(t) (A + G) \\
- \Pi(t) BR^{-1} B^T \Pi(t) + (I - \Gamma)^T Q (I - \Gamma).
\]
with \( \Pi(T) = 0 \). By Sun et al. (2016, Theorem 4.5), the solvability of (10) and (11) is equivalent to the uniform convexity of two optimal control problems. Particularly, if \( Q \geq 0 \), then (10) and (11) admit a unique solution, respectively.

\[\text{Theorem 3.2.} \text{ Assume (A1) holds, and (10)--(11) admit a solution, respectively. Then (P1) has an optimal control} \]
\[
\hat{u}_i(t) = -R^{-1} B^T [P(t) \dot{x}_i(t) + K(t) \dot{x}_N(t) + s(t)], \quad i = 1, \ldots, N.
\]
where \( P, K \) and \( s \) are determined by (10)–(12).

To prove Theorem 3.2, we first provide a lemma, which plays a key role in the later analysis.

\[\text{Lemma 3.1.} \text{ If (10) and (11) admit a solution, respectively, then Problem (P1) is uniformly convex.} \]

\[\text{Proof.} \text{ By (10), (13), and direct calculations, we have} \]
\[
\sum_{i=1}^{N} E \int_0^T e^{-\mu t} \left( \left\| y_i(t) - \Gamma y_i^N(t) \right\|_Q^2 + \left\| u_i(t) \right\|_R^2 \right) dt
\]
\[
= \sum_{i=1}^{N} E \int_0^T e^{-\mu t} \left( \left\| y_i(t) \right\|_Q^2 - \left\| y_i(t) \right\|_Q^2 + \left\| u_i(t) \right\|_R^2 \right) dt
\]
\[
= \sum_{i=1}^{N} E \int_0^T e^{-\mu t} \left( \left\| y_i(t) - \dot{y}_i^N(t) \right\|_Q^2 + \left\| u_i(t) \right\|_R^2 \right) dt
\]
\[
= \sum_{i=1}^{N} E \int_0^T e^{-\mu t} \left( \left\| u_i(t) \right\|_R^2 + \left\| u_i(t) \right\|_R^2 \right) dt
\]
\[
+ \left\| u_i(t) \right\|_R^2 + \left\| u_i(t) \right\|_R^2 \right) dt
\]
Proof of Theorem 3.2. Since (10) and (11) have a solution, respectively, then by Ma and Yong (1999, Chapter 2, §4), (9) admits a unique solution. Thus, the FBSDE (5) is decoupled and no fixed-point equation is needed. Here, we first obtain the centralized open-loop solution by variational analysis. By tackling the coupled FBSDE (10), respectively. The set of decentralized control laws is obtained by (10)–(12). □

Remark 3.3. Here, we firstly obtain the centralized open-loop solution by variational analysis. By tackling the coupled FBSDE (10), respectively. The set of decentralized control laws is obtained by (10)–(12). □

Theorem 3.3. Assume that (A1) holds, and (10)–(11) admit a solution, respectively. The set of decentralized control laws $\bar{u}_1, \ldots, \bar{u}_N$ in (15) has asymptotic social optimality, i.e.,

$$\left|\frac{1}{N} f_{\text{soc}}(\bar{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}(0,T,K)} f_{\text{soc}}(u)\right| = O\left(\frac{1}{\sqrt{N}}\right),$$

and the corresponding social cost is given by

$$f_{\text{soc}}(\bar{u}) = \sum_{i=1}^{N} \mathbb{E}\left[\left\|x(0,\bar{u}) - x(N,0)\right\|_{\mathcal{F}}^{2} + \left\|x(N,0)\right\|_{\mathcal{F}}^{2} + 2\mathbb{E}\left[\mathcal{X}(0,\bar{u})\right] + Nq_{T} + Ne_{T},$$

where

$$q_{T} = \int_{0}^{T} e^{-\rho t} \left[\left\|\sigma(t)\right\|_{\mathcal{H}}^{2} + \left\|\sigma(t)\right\|_{\mathcal{F}}^{2}\right] dt,$$

$$e_{T} = \mathbb{E}\left[\int_{0}^{T} e^{-\rho t} \left\|\mathcal{X}(t)\right\|_{\mathcal{F}}^{2} dt\right].$$

Proof. See Appendix A. □

4. The infinite-horizon problem

Based on the analysis in Section 3, we may design the following decentralized control laws for Problem (P):

$$\hat{u}_i(t) = -R^{-1}B^{i}[\hat{P}\hat{x}(t) + (\bar{P} - P)\hat{x}(t) + s(t)],$$

where $P$ and $\bar{P}$ are maximal solutions\(^1\) to the equations

$$\rho P = A^{\top}P + PA - PBR^{-1}B^{i}P + Q,$$

$$\rho \bar{P} = (A+G)^{\top} \bar{P} + \bar{P}(A+G) - \bar{P}BR^{-1}B^{i} \bar{P} + Q - \Xi,$$

and $s(0)$ is to be determined, and the existence conditions of $P, \bar{P}, s$ need to be investigated further.

4.1. Uniform stabilization of subsystems

We now list some basic assumptions for reference:

(A2) The system $(A - \frac{\delta}{2}I, B)$ is stabilizable, and $(A + G - \frac{\delta}{2}I, B)$ is Hurwitz, where $\delta \triangleq A - BR^{-1}B^{i}P$.

(A3) $Q > 0, (A-\frac{\delta}{2}I, \sqrt{\Xi})$ is detectable, and $(A+G-\frac{\delta}{2}I, \sqrt{\Xi}(I-\Gamma))$ is observable.

Assumptions (A2) and (A3) are basic in the study of the LQ optimal control problem. We will show that under some conditions, (A2) is also necessary for uniform stabilization of multiagent systems. In many cases, (A3) may be weakened to the following assumption.

(A3') $Q > 0, (A-\frac{\delta}{2}I, \sqrt{\Xi})$ is detectable, and $(A+G-\frac{\delta}{2}I, \sqrt{\Xi}(I-\Gamma))$ is observable.

Lemma 4.1. Under (A2)–(A3), (21) and (22) admit unique solutions $P > 0, \bar{P} > 0$, respectively, and (23)–(24) admits a set of unique solutions $x, \hat{x} \in C_{\mathcal{F}}([0,\infty), \mathbb{R}^n)$.

Proof. From (A2)–(A3) and (Anderson & Moore, 1990), (21) and (22) admit unique solutions $P > 0, \bar{P} > 0$ such that $A - BR^{-1}B^{i}P - \frac{\delta}{2}I$ and $A+G - BR^{-1}B^{i} \bar{P} - \frac{\delta}{2}I$ are Hurwitz, respectively. From an argument in Wang and Zhang (2012a, Appendix A), we obtain $s(0) \in C_{\mathcal{F}}([0,\infty), \mathbb{R}^n)$ if and only if

$$s(t) = \int_{t}^{\infty} e^{-\rho s} \left[\mathcal{X}(s) + \hat{P}\hat{x}(s) + (\bar{P} - P)\hat{x}(s)\right] ds.$$

Lemma 4.2. Let (A1)–(A3) hold. Then for Problem (P),

$$\mathbb{E}\int_{0}^{\infty} e^{-\rho t} \left\|\mathcal{X}(t) - \hat{x}(t)\right\|_{\mathcal{F}}^{2} dt = O\left(\frac{1}{\sqrt{N}}\right),$$

where $\hat{x}(N) = \sum_{i=1}^{N} \hat{x}_i$, and $\hat{x}$ satisfies (24).

Proof. See Appendix B. □

It is shown that the decentralized control laws (15) uniformly stabilize the systems (1).\(^1\)

\(^1\) For a Riccati equation (e.g., (21)), $P$ is called a maximal solution if for any solutions $P', P - P' \geq 0$. 

Let (A1)–(A3) hold. Then for any N,
\[ \sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 \right) dt < \infty. \] (26)

**Proof.** See Appendix B. □

We now give two equivalent conditions for uniform stabilization of multiagent systems.

**Theorem 4.2.** Let (A3) hold. Assume that (21)–(22) admit symmetric solutions. Then for Problem (P) the following statements are equivalent:

(i) For any initial condition \((\hat{x}_1(0), \ldots, \hat{x}_N(0))\) satisfying (A1),
\[ \sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 \right) dt < \infty. \] (27)

(ii) Eqs. (21) and (22) admit unique maximal solutions such that \(P > 0, \Pi > 0\), and \(A + G - \frac{\rho}{2} I\) is Hurwitz.

(iii) (A2) holds.

**Proof.** See Appendix C. □

For \(G = 0\), we have a simplified version of Theorem 4.2.

**Corollary 1.** Assume that (A3) holds and \(G = 0\). Assume that (21)–(22) admit symmetric solutions. Then the following statements are equivalent:

(i) For any \((\hat{x}_1(0), \ldots, \hat{x}_N(0))\) satisfying (A1),
\[ \sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 \right) dt < \infty. \]

(ii) Eqs. (21) and (22) admit unique maximal solutions such that \(P > 0, \Pi > 0\), respectively.

(iii) The system \((A - \frac{\rho}{2} I, B)\) is stabilizable.

When (A3) is weakened to (A3’), we have the following equivalent conditions of uniform stabilization.

**Theorem 4.3.** Let (A3’) hold. Assume that (21)–(22) admit solutions. Then the following are equivalent:

(i) For any initial condition \((\hat{x}_1(0), \ldots, \hat{x}_N(0))\) satisfying (A1),
\[ \sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 \right) dt < \infty. \]

(ii) Eqs. (21) and (22) admit unique maximal solutions \(P \geq 0, \Pi \geq 0\), and \(A + G - \frac{\rho}{2} I\) is Hurwitz.

(iii) (A2) holds.

**Proof.** See Appendix C. □

For the more general case that \(Q\) are indefinite, we have the following equivalent conditions for uniform stabilization of all the subsystems. Assume \((A3’’)\) both \(M_1\) and \(M_2\) have no eigenvalues on the imaginary axis, where

\[ M_1 = \begin{bmatrix} A - \frac{\rho}{2} I & BR^{-1}B^T \\ Q & -A^T + \frac{\rho}{2} I \end{bmatrix}, \]

\[ M_2 = \begin{bmatrix} A + G - \frac{\rho}{2} I & BR^{-1}B^T \\ Q - \Sigma & -(A + G)^T + \frac{\rho}{2} I \end{bmatrix}. \]

**Theorem 4.4.** Assume that \((A3’’')\) holds, and (21)–(22) admit solutions. Then the following are equivalent:

(i) For any \((\hat{x}_1(0), \ldots, \hat{x}_N(0))\) satisfying (A1),
\[ \sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 \right) dt < \infty. \]

(ii) Eqs. (21) and (22) admit unique \(\rho\)-stabilizing solutions\(^2\) (which are also the maximal solutions), and \(A + G - \frac{\rho}{2} I\) is Hurwitz.

(iii) (A2) holds.

\(^2\) For a Riccati equation (21), \(P\) is called a \(\rho\)-stabilizing solution if \(P\) satisfies (21) and all the eigenvalues of \(A - BR^{-1}B^T P - \frac{\rho}{2} I\) are in left-half-plane.
Proof. Social optimality, i.e., Assume that there exists a solution such that Let \( q \neq 0 \) then (29) has a unique positive solution such that \( a - b^2p/\rho - \rho/2 = -\sqrt{\Delta}/2 < 0 \). If \( q = 0 \) and \( a - \rho/2 < 0 \) then (30) has a unique non-negative solution \( p = 0 \) such that \( a - b^2p/\rho - \rho/2 = a - \rho/2 < 0 \).

Assume that (28) and (29) hold. By Theorem 4.4, the system is uniformly stable if and only if \((a - \rho/2, b)\) is stabilizable (i.e., \( b \neq 0 \) or \( a - \rho/2 < 0 \), and \( a - b^2p/\rho - \rho/2 + g < 0 \). Note that \( a - b^2p/\rho - \rho/2 + g < 0 \). When \( g \leq 0 \), we have \( a - b^2p/\rho - \rho/2 + g < 0 \).

Example 2. We further consider the model in Example 1 for the case that \( a + g = \rho/2 \) and \( \gamma = 1 \) (i.e., (29) does not hold). In this case, the Riccati equation (22) admits a unique solution \( H_t = 0 \). (23) becomes \( \dot{\rho}(t) = \tilde{s}(t) + \frac{\tilde{\xi}}{2}(t) \) and has a unique solution \( \tilde{\xi}(t) \equiv 0 \) in \( C_{\rho/\delta}([0, \infty), \mathbb{R}) \). Thus, \( \tilde{\xi} \) satisfies

\[
\frac{d\tilde{\xi}}{dt} = \frac{\rho}{2} \tilde{\xi}(t) + f(t). \tag{31}
\]

Assume that \( f \) is a constant. Then (31) does not admit a solution in \( C_{\rho/\delta}([0, \infty), \mathbb{R}) \) unless \( \tilde{\xi}(0) = -2f/\rho \).

4.2. Asymptotic social optimality

Now we are in a position to state the asymptotic optimality of the decentralized control.

Theorem 4.5. Let (A1)-(A3) hold. For Problem (P), the set of decentralized control laws \( \{\tilde{u}_1, \ldots, \tilde{u}_N\} \) given by (20) has asymptotic social optimality, i.e.,

\[
\frac{1}{N} J_{soc}(\tilde{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}} J_{soc}(u) = O(1/\sqrt{N}).
\]

Proof. We first prove that for \( u \in \mathcal{U} \), \( J_{soc}(u) < NC_1 \) implies that

\[
\sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} (\|x_i(t)\|^2 + \|u_i(t)\|^2) dt < NC_2,
\]

for all \( i = 1, \ldots, N \). From \( J_{soc}(u) < NC_1 \), we have \( \sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} (\|u_i(t)\|^2) dt < NC \)

\[
\sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \|\dot{x}(t) - \Gamma x(t)\|_Q^2 dt < NC,
\]

which further implies that

\[
\mathbb{E} \int_{0}^{\infty} e^{-\rho t} \|I - \Gamma\|_{Q} x(t)\|_Q^2 dt < C.
\]

By (1) we have

\[
\mathbb{E} \int_{0}^{\infty} e^{-\rho t} \|x(t)\|_Q^2 dt < C.
\]

which leads to for any \( r \in [0, 1], \)

\[
\chi^N(t) = e^{(A + G)\gamma x N(t - r)} + \int_{t-r}^{t} e^{(A + G)(t - \tau)} [Bu^N(\tau) + f(\tau)] d\tau \tag{35}
\]

which implies (30) admits two solutions. If \( q > 0 \) then (30) has a unique positive solution such that \( a - b^2p/\rho - \rho/2 = -\sqrt{\Delta}/2 < 0 \). If \( q = 0 \) and \( a - \rho/2 < 0 \) then (30) has a unique non-negative solution \( p = 0 \) such that \( a - b^2p/\rho - \rho/2 = a - \rho/2 < 0 \).

By \( J_{soc}(u) < C_1 \) and basic SDE estimates, we can find a constant \( C \) such that

\[
\mathbb{E} \int_{t}^{\infty} e^{-\rho t} \left\| \int_{t-r}^{t} e^{(A + G)(t - \tau)} Bu^N(\tau) d\tau \right\|^2 dt < C.
\]

From (34) and (35) we obtain

\[
\mathbb{E} \int_{t}^{\infty} e^{-\rho t} \left\| \int_{t-r}^{t} e^{(A + G)(t - \tau)} G(\tau) Q(1 - I^{-1}) \right\| d\tau < C,
\]

which implies that for any \( r \in [0, 1] \),

\[
\mathbb{E} \int_{t}^{\infty} e^{-\rho t} \left\| \int_{t-r}^{t} e^{(A + G)(t - \tau)} G(\tau) Q(1 - I^{-1}) \right\| d\tau < C,
\]

By taking integration with respect to \( r \in [0, 1] \), we obtain

\[
\mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left\| \int_{0}^{t} e^{(A + G)(t - \tau)} G(\tau) Q(1 - I^{-1}) \right\| d\tau < C.
\]

This together with (A3) leads to

\[
\int_{0}^{\infty} e^{-\rho t} \left\| \int_{0}^{t} e^{(A + G)(t - \tau)} G(\tau) Q(1 - I^{-1}) \right\| d\tau < C,
\]

which with (33) further gives

\[
\sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \|x_i(t)\|^2 dt < NC.
\]

By (1), we have

\[
\chi(t) = e^{\rho t} \chi(t - r) + \int_{t-r}^{t} e^{(A + G)(t - \tau)} [Bu(\tau) + f(\tau) + G\chi(t - r)] d\tau
\]

It follows from (36) that

\[
\mathbb{E} \int_{t}^{\infty} e^{-\rho t} \left\| \int_{t-r}^{t} e^{(A - \chi N(t))} d\tau \right\|^2 dt \leq \mathbb{E} \int_{t}^{\infty} e^{-\rho t} \left\| G(\tau) Q(1 - I^{-1}) \right\|^2 dt \leq C.
\]

From (37) and (38), we obtain that

\[
\sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \|x_i(t)\|^2 dt < NC,
\]

which gives (32). From this with Theorem 4.1.
By a similar argument to the proof of Theorem 3.3 combined with Lemma 4.2, the conclusion follows. □

If (A3) is replaced by (A3'), the decentralized control (20) still has asymptotic social optimality.

**Corollary 2.** Assume that (A1)–(A2), (A3') hold. The decentralized control (20) is asymptotically social optimal.

**Proof.** Without loss of generality, we simply assume $A + G = \text{diag}(A_1, A_2)$, where $A_1 - (\rho/2)I$ is Hurwitz, and $-A_2 - (\rho/2)I$ is Hurwitz (if necessary, we may apply a nonsingular linear transformation as in the proof of Theorem 4.3). Write $x^{(N)} = \left[ z_1^T, z_2^T \right]^T$ and $Q^{1/2}(I - \Gamma) = \left[ S_1, S_2 \right]$ such that \( \| (I - \Gamma) x^{(N)}(t) \|_Q^2 = \| S_1 z_1(t) + S_2 z_2(t) \|_2^2 \), and $A_2 - (\rho/2) I, S_2$ is observable. By the proof of Theorem 4.1 or (Huang, 2010), $E \int_0^\infty e^{-\rho t} \| u^{(N)}(t) \|^2 dt < \infty$ implies $E \int_0^\infty e^{-\rho t} \| z_1(t) \|^2 dt < \infty$, which together with (34) gives $E \int_0^\infty e^{-\rho t} \| S_2 z_2(t) \|^2 dt < \infty$. This and the observability of $(A_2 - (\rho/2) I, S_2)$ leads to $E \int_0^\infty e^{-\rho t} \| S_2 z_2(t) \|^2 dt < \infty$. Thus, $E \int_0^\infty e^{-\rho t} \| x^{(N)}(t) \|^2 dt < \infty$. The other parts of the proof are similar to that of Theorem 4.5. □

For the case that $Q$ are indefinite, we have the following result of asymptotic optimality.

**Theorem 4.6.** Let (A1)–(A2), (A3') hold. Assume (21)–(22) admit negative definite solutions $P^- < 0$ and $\Pi^- < 0$, respectively. Then, the set of decentralized control (20) is asymptotically social optimal. Furthermore, if $\{ x_0 \}$ have the same variance, then the asymptotic average social optimum is given by

$$
\lim_{N \to \infty} \frac{1}{N} J_{soc}(\hat{u}) = E \left[ \| x_0 - \hat{x}_0 \|^2_F + \| \hat{x}_0 \|^2_F + 2s^T(0) \hat{x}_0 \right] + q_\infty,
$$

where

$$
q_\infty = \int_0^\infty e^{-\rho t} \left[ \| \sigma(t) \|_2^2 + \| \sigma(T) \|_2^2 \right] - \| B^T s(t) \|^2_{F-1} + 2s^T(T \sigma(T)) dt.
$$

**Proof.** From the above assumptions and Theorem 4.4, the Riccati equation (21) admits a $\rho$-stabilizing solution $P$ and a negative definite solution $P^-$; (22) has a $\rho$-stabilizing solution $\Pi$ and a negative definite solution $\Pi^-$. By a similar argument in the proof of Lemma 3.1, we obtain for any $u \in U_c$,

$$
J_{soc}(u) = \sum_{i=1}^N E \left[ \int_0^\infty e^{-\rho t} \left( \| x_i - x_i(0) \|^2_Q + \| x_i^{(N)}(0) \|^2_{Q-\epsilon} + \| u_i \|^2_Q \right) - 2\eta_i Q(I - \Gamma) x_i + \| u_i - u_i^{(N)} \|^2_R + \| u_i^{(N)} \|^2_R \right] dt,
$$

$$
= \sum_{i=1}^N E \left[ \| x_i(0) \|^2_{F^-} + \| x_i^{(N)}(0) \|^2_{F^-} \right] + 2s^T(0) x_i - \sum_{i=1}^N \int_0^\infty e^{-\rho t} E \left[ \| x_i^{(N)}(T) \|^2_{F^-} \right] dt
$$

$$
+ \| x_i(T) - x_i^{(N)}(T) \|^2_R + 2s^T(T x_i^{(N)}(T))
$$

$$
+ \sum_{i=1}^N E \left[ \int_0^\infty e^{-\rho t} \left( \| u_i^{(N)} + R^{-1}B^T \Pi x_i^{(N)} \|^2_R + \| u_i - u_i^{(N)} + R^{-1}B^T P^-(x_i - x_i^{(N)}) \|^2_R \right) dt \right] + q_\infty.
$$

By Willems (1971, Theorem 8), the centralized optimal control exists and the optimal state is $\rho$-stable. Thus, we only need to consider the following control set

$$
U_c = \left\{ (u_1, \ldots, u_N) | u_i(t) \text{ is adapted to } F_i, \quad E \int_0^\infty e^{-\rho t} \| x_i(t) \|^2 dt < \infty, \forall i \right\}.
$$

For any $u \in U_c$ satisfying $J_{soc}(u) \leq NC$, we have

$$
J_{soc}(u) = \sum_{i=1}^N E \left[ \| x_i(0) \|^2_{F^-} + \| x_i^{(N)}(0) \|^2_{F^-} + 2s^T(0) x_i \right]
$$

$$
+ \sum_{i=1}^N E \left[ \int_0^\infty e^{-\rho t} \left( \| u_i - u_i^{(N)} + R^{-1}B^T P^-(x_i - x_i^{(N)}) \|^2_R \right) dt \right] + q_\infty \leq NC.
$$

Denote $\nu^{(N)} = u^{(N)} + R^{-1}B^T \Pi x^{(N)}$. From (1),

$$
dx^{(N)}(t) = (A + G - BR^{-1}B^T \Pi) x^{(N)}(t) dt + B \nu^{(N)}(t) dt + \frac{1}{N} \sum_{i=1}^N \sigma_i(t) dW_i(t).
$$

By Huang (2010), there exist $C_1, C_2 > 0$ such that

$$
E \int_0^\infty e^{-\rho t} \| x^{(N)}(t) \|^2 dt \leq C_1 E \int_0^\infty e^{-\rho t} \| u^{(N)}(t) \|^2 dt + C_2.
$$

This together with (40) gives

$$
\sum_{i=1}^N E \int_0^\infty e^{-\rho t} \left( \| x_i^{(N)}(t) \|^2 + \| u_i^{(N)}(t) \|^2 \right) dt
$$

$$
= N E \int_0^\infty e^{-\rho t} \left( \| x_i(t) - x_i^{(N)}(t) \|^2 + \| u_i(t) - u_i^{(N)}(t) \|^2 \right) dt
$$

$$
\leq NC_1 E \int_0^\infty e^{-\rho t} \| x^{(N)}(t) \|^2 + NC_4 \leq NC.
$$

Similarly, we have

$$
\sum_{i=1}^N E \int_0^\infty e^{-\rho t} \left( \| x_i(t) - x_i^{(N)}(t) \|^2 + \| u_i(t) - u_i^{(N)}(t) \|^2 \right) dt \leq NC.
$$

The remainder of the proof can follow by that of Theorem 3.3. For the case that $\{ x_0 \}$ have the same variance, from (17), the asymptotic average social optimum ($\lim_{N \to \infty} \frac{1}{N} J_{soc}(\hat{u})$) is given by

$$
E \left[ \| x_0 - \hat{x}_0 \|^2_F + \| \hat{x}_0 \|^2_F + 2s^T(0) \hat{x}_0 \right] + q_\infty. \quad \square
$$

**Remark 4.3.** The work Huang et al. (2012) investigated mean field IQ problem (P) with $Q \geq 0$. To obtain asymptotic social optimality, they need $Q > 0$ and $I - \Gamma$ is nonsingular. In Corollary 2, we have loosened the assumption to (A3'), i.e., $(A - (\rho/2)I, \sqrt{Q})$ and $(A - (\rho/2)I, \sqrt{Q}(I - \Gamma))$ are detectable. In Theorem 4.6, we further give the condition for the case of indefinite $Q$. Particularly, for the scalar case, the condition is equivalent to (28)–(29). It can be verified that the assumption $Q > 0$ and $I - \Gamma$ is nonsingular implies (28)–(29), but the converse is not true.

4.3. Comparison to previous solutions

In this section, we compare the proposed decentralized control laws with the feedback decentralized strategies in previous works.
We first introduce a definition from Basar and Olsder (1982).

**Definition 4.1.** For a control problem with an admissible control set \( \mathcal{U} \), a control law \( u \in \mathcal{U} \) is said to be a representation of another control \( u^* \in \mathcal{U} \) if
(i) they both generate the same unique state trajectory, and
(ii) they both have the same open-loop value on this trajectory.

For Problem \( (P) \), let \( f = 0 \), and \( G = 0 \). In Huang et al. (2012, Theorem 4.3), the decentralized control laws are given by
\[
\hat{u}_i(t) = -R^{-1}B^T(P\hat{x}_i(t) + \bar{s}(t)), \quad i = 1, \ldots, N,
\]
where \( P \) is the stabilizing solution of \( (21) \), and \( \bar{s} = \bar{K}x^l + \phi \). Here \( \bar{K} \) satisfies
\[
\rho\bar{K} = \bar{K}A + \bar{A}^T\bar{K} - \bar{K}BR^{-1}B^T\bar{K} - \bar{Z},
\]
and \( x^l, \phi \in C_{\rho/2}[0, \infty), \mathbb{R}^n \) are given by
\[
\begin{align*}
\frac{dx^l}{dt} &= \bar{A}x^l(t) - BR^{-1}B^T(\bar{K}x^l(t) + \phi(t)), \quad x^l(0) = \bar{x}_0, \\
\frac{d\phi}{dt} &= -[A - BR^{-1}B^T(P + \bar{K}) - \rho I]\phi(t) + \bar{\eta}(t),
\end{align*}
\]
in which \( \bar{A} = A - BR^{-1}BP \) and \( \phi(0) \) is to be determined by \( \phi \in C_{\rho/2}[0, \infty), \mathbb{R}^n \). By comparing this with \( (22)-(24) \), one can obtain that \( \bar{K} = P - P \), \( \bar{x} = \bar{x}^l \) and \( \phi = s \). From the above discussion, we have the equivalence of the two sets of decentralized control laws.

**Proposition 4.1.** The set of decentralized control laws \( \{\hat{u}_1, \ldots, \hat{u}_N\} \) in \( (20) \) is a representation of \( \{\bar{u}_1, \ldots, \bar{u}_N\} \) given by \( (42) \).

**Remark 4.4.** The work Huang et al. (2012) studied the problem \( (P) \) with \( Q \geq 0 \) by the fixed-point approach. In Theorem 4.3, they have shown that the fixed-point equation admits a unique solution, when \( (A - (\rho/2)I, \sqrt{Q}) \) is detectable and \( \Xi = \Gamma^TQ + Q\Gamma^T - \rho \Gamma \leq 0 \). In fact, the above assumption is merely a sufficient condition to ensure \( (A^3') (A - \rho I), \sqrt{Q - \Xi} \) is detectable.

**Remark 4.5.** The work Huang and Zhou (2020) investigated asymptotic solvability of mean field LQ games by the re-scaling method. They considered \( (1)-(2) \) with \( Q \geq 0 \) and derived a low-dimensional ordinary differential equation system by dynamic programming. Actually, the method proposed in this paper can be viewed as a type of direct approach. Different from Huang and Zhou (2020), we tackle directly high-dimensional FBSDEs along the line of maximum principle.

5. Numerical examples

Now, two numerical examples are given to illustrate the effectiveness of the proposed decentralized control.

We first consider a scalar system with 30 agents in Problem \( (P) \). Take \( A = 0.8, B = R = 1, Q = -0.1, G = -0.2, f(t) = 1, \sigma(t) = 0.2, \rho = 0.6, \Gamma = 0.2, \eta = 5 \) in \( (1)-(2) \). The initial states of 50 agents are taken independently from a normal distribution \( N(0, 0.3) \). Note that \( B \neq 0 \), and \( \bar{A} + G - \frac{1}{2}I = -0.5873 < 0 \). Then \( (A1)-(A2) \) hold. Since \( M_1 = \begin{bmatrix} 0.5 & 1 \\ -0.1 & -0.5 \end{bmatrix} \) and \( M_2 = \begin{bmatrix} 0.3 & 0 \\ -0.064 & -0.3 \end{bmatrix} \) have no eigenvalues on the imaginary axis, \( (A^3') \) also holds. Under the control law \( (20) \), the trajectories of \( \bar{x} \) and \( \bar{x}^{(N)} \) in Problem \( (P) \) are shown in Fig. 1. It can be seen that \( \bar{x} \) and \( \bar{x}^{(N)} \) coincide well, which illustrate the consistency of mean field approximations.

Denote \( \epsilon = \frac{1}{n_{soc}} \inf_{u \in \mathcal{U}_{soc}} J_{soc}(u) \). By Theorems 3.3 and 4.6, we obtain \( \epsilon = \int_0^\infty e^{-\tau} \|B^T\bar{K}(\bar{x}^{(N)}(t) - \bar{x}(t))\|^2_{\Sigma} dt \). The cost gap \( \epsilon \) is demonstrated in Fig. 2 where the agent number \( N \) grows from 1 to 200.

Finally, we consider the 2-dimensional case of Problem \( (P) \).

Take parameters as follows: \( A = \begin{bmatrix} 0.1 & 0 \\ -1 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, G = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.3 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \eta = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, f = [1, 1]^T \) and \( \sigma = [0.5, 0.5]^T \). Denote \( \hat{x}_i(t) = [\hat{x}_i^l(t), \hat{x}_i^r(t)]^T \). Both of \( \hat{x}_i^l(0) \) and \( \hat{x}_i^r(0) \) are taken independently from a normal distribution \( N(5, 0.5) \). Under the control laws \( (20) \), the trajectories of \( \hat{x}_i^l \) and \( \hat{x}_i^r, i = 1, \ldots, N \) are shown in Figs. 3 and 4, respectively. The curves of \( \hat{x}_i^l, i = 1, \ldots, 30 \) soon converge to 0 with some fluctuation. The curves of \( \hat{x}_i^r, i = 1, \ldots, 30 \) first decrease and then grow up before the time 40. After a period of time, they converge to a constant, which verify the \( \rho \)-stability of agents.

6. Concluding remarks

In this paper, we have considered uniform stabilization and social optimality for mean field LQ multiagent systems. For finite- and infinite-horizon problems, we design the decentralized control laws by decoupling FBSDEs, respectively, which are further shown to be asymptotically optimal. Some equivalent conditions are further given for uniform stabilization of the systems in
different cases. Finally, we compare such decentralized control laws with the asymptotic optimal strategies in previous works.

An interesting generalization is to consider mean field LQ control systems with heterogeneous coefficients by the direct approach (He et al., 2015). Also, the variational analysis may be applied to construct decentralized control laws for the nonlinear social control model.

Appendix A. Proof of Theorem 3.3

To prove Theorem 3.3, we need a lemma.

Lemma A.1. Let (A1) hold. Assume that (10) and (11) admit a solution, respectively. Under the control (15), we have

$$\max_{0 \leq t \leq T} \mathbb{E}\|\hat{x}^N(t) - \bar{x}(t)\|^2 = O(1/N).$$

(A.1)

Proof. It follows by (16) that

$$d\hat{x}^N(t) = \left[ (\bar{A}(t) + G)\hat{x}^N(t) - BR^{-1}B^T(K(t)\bar{x}(t) + s(t)) + f(t) \right] dt + \frac{1}{N} \sum_{i=1}^{N} \sigma(t) d\mathcal{W}_i(t).$$

where $\bar{A}(t) = A - BR^{-1}B^T P(t).$ From (14), we have

$$\hat{x}^N(t) - \bar{x}(t) = \Phi(t)|\hat{x}^N(0) - \bar{x}(0)| + \frac{1}{N} \sum_{i=1}^{N} \int_0^t \Phi(t, \tau) \sigma(\tau) d\mathcal{W}_i(\tau).$$

(A.2)

where $\Phi$ satisfies $\frac{d}{dt} \Phi(t, \tau) = (\bar{A}(t) + G)\Phi(t, \tau), \quad \Phi(\tau, \tau) = I.$ By (A1), one can obtain

$$\mathbb{E}\|\hat{x}^N(t) - \bar{x}(t)\|^2 \leq 2\|\Phi(t, 0)\|^2 \left\{ \mathbb{E}\|\hat{x}^N(0) - \bar{x}(0)\|^2 + \frac{2}{N} \int_0^t \|\Phi(t, \tau)\|\sigma(\tau)\|^2 d\tau \right\} \leq 2\|\Phi(t, 0)\|^2 \max_{1 \leq \tau \leq N} \mathbb{E}\|\hat{x}_0\|^2 + \frac{2}{N} \int_0^T \|\Phi(t, \tau)\|^2 \|\sigma(\tau)\|^2 d\tau,$$

(A.3)

which completes the proof. □

Proof of Theorem 3.3. Note that $\inf_{u \in L^2_{loc}(0, T; \mathbb{R}^m)} f_{soc}(u) \leq f_{soc}(\hat{u}).$

To obtain

$$\frac{1}{N} f_{soc}(\hat{u}) \leq \frac{1}{N} f_{soc}(u) + O\left(\frac{1}{\sqrt{N}}\right),$$

we only need to prove for any $u \in \mathcal{U}^\prime \equiv \{ u \in L^2_{loc}(0, T; \mathbb{R}^m) : f_{soc}(u) \leq f_{soc}(\hat{u}) \}$, the following holds:

$$\frac{1}{N} f_{soc}(\hat{u}) \leq \frac{1}{N} f_{soc}(u) + O\left(\frac{1}{\sqrt{N}}\right).$$

We now show that for $u \in \mathcal{U}^\prime$, $\sum_{i=1}^{N} \mathbb{E}\int_0^T e^{-\rho t} (\|x_i(t)\|^2 + \|u_i(t)\|^2) dt < NC_2.$ By Lemma 3.1, (P1) is uniformly convex which gives there exists $\delta_0 > 0$ such that

$$\delta_0 \sum_{i=1}^{N} \mathbb{E}\int_0^T e^{-\rho t} \|u_i(t)\|^2 dt - C \leq f_{soc}(u).$$

Since $f_{soc}(\hat{u}) < NC_1$, we have $f_{soc}(u) < NC_1$, which implies $\sum_{i=1}^{N} \mathbb{E}\int_0^T e^{-\rho t} \|u_i(t)\|^2 dt < NC.$ This leads to

$$\mathbb{E}\int_0^T e^{-\rho t} \|u(t)\|^2 dt \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\int_0^T e^{-\rho t} \|u_i(t)\|^2 dt < C,$$

where $u(t) = \frac{1}{N} \sum_{i=1}^{N} u_i.$ By (1).

$$d\hat{x}^N(t) = \left[ (A + G)\hat{x}^N(t) + Bu^N(t) + f(t) \right] dt + \frac{1}{N} \sum_{i=1}^{N} \sigma(\tau) d\mathcal{W}_i(t),$$

which implies $\max_{0 \leq t \leq T} \mathbb{E}\|\hat{x}^N(t)\|^2 \leq C.$ Note that

$$x_i(t) = e^{At} x_{i0} + \int_0^t e^{A(t-\tau)} \sigma(\tau) d\mathcal{W}_i(\tau) + \int_0^t e^{A(t-\tau)} [G\hat{x}^N(\tau) + Bu_i(\tau) + f(\tau)] d\tau.$$
We have
\[ N \sum_{i=1}^{N} E \int_{0}^{T} e^{-\rho t} \|\mathbf{x}_i(t)\|^2 dt \leq C \left( \sum_{i=1}^{N} E \|\mathbf{x}_0\|^2 + N \max_{0 \leq t \leq T} E \|\dot{\mathbf{x}}(t)\|^2 \right) + N \sum_{i=1}^{N} E \int_{0}^{T} e^{-\rho t} \|u_i(t)\|^2 dt + NC_1 ) < NC_2. \] (A.4)

By (14) and (16), we obtain that
\[ E \int_{0}^{T} e^{-\rho t} (\|\dot{\mathbf{x}}_i(t)\|^2 + \|\ddot{\mathbf{x}}_i(t)\|^2 + \|\dddot{\mathbf{x}}(t)\|^2) dt < C. \] (A.5)

Let \( \tilde{\mathbf{x}}_i = \mathbf{x}_i - \xi_i, \tilde{u}_i = u_i - \tilde{u}_i \) and \( \tilde{\mathbf{x}}(n) = \frac{1}{N} \sum_{i=1}^{N} \tilde{\mathbf{x}}_i \). Then by (1) and (16),
\[ d\tilde{\mathbf{x}}_i(t) = \left( A\tilde{\mathbf{x}}_i(t) + G\tilde{\mathbf{x}}_i(t) + B\tilde{u}_i(t) \right) dt, \quad \tilde{\mathbf{x}}_i(0) = 0. \] (A.6)

From (3), \( J^{(\tilde{u})} = \sum_{i=1}^{N} (J^{(u)}(\tilde{u}) + \int_{0}^{T} \tilde{I}_i dt + \int_{0}^{T} \tilde{I}_i dt) \)
where
\[ \tilde{I}_i = 2 \mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ \|\tilde{\mathbf{x}}_i(t) - \Gamma\tilde{\mathbf{x}}(t) - \eta\|_Q \right] \frac{1}{2} \right] \left[ \|\tilde{u}_i(t)\|_R^2 \right] dt. \]

By Lemma 3.1 and Proposition 3.1, \( J^{(\tilde{u})} \geq 0 \). We only need to prove \( \frac{1}{N} \sum_{i=1}^{N} \tilde{I}_i = O\left(\frac{1}{\sqrt{N}}\right) \). By direct computations, one can obtain
\[ N \sum_{i=1}^{N} \tilde{I}_i = \sum_{i=1}^{N} \left[ 2 \mathbb{E} \int_{0}^{T} \int_{0}^{T} e^{-\rho t} \left[ \mathbb{E} \left( \tilde{\mathbf{x}}_i(t) - \Gamma\tilde{\mathbf{x}}(t) - \eta \right) \right] \right] \left[ \|\tilde{u}_i(t)\|_R^2 \right] dt \]
\[ + N \mathbb{E} \int_{0}^{T} e^{-\rho t} (\tilde{\mathbf{x}}(n) - \tilde{\mathbf{x}}(t)) \frac{T}{2} \right] \left[ \|\tilde{\mathbf{x}}_i(t) - \Gamma\tilde{\mathbf{x}}(t) - \eta\|_Q \right] \frac{1}{2} \right] \left[ \|\tilde{u}_i(t)\|_R^2 \right] dt \]
\[ + N \mathbb{E} \int_{0}^{T} e^{-\rho t} (\tilde{\mathbf{x}}(n) - \tilde{\mathbf{x}}(t)) \frac{T}{2} \right] \left[ \|\tilde{\mathbf{x}}_i(t) - \Gamma\tilde{\mathbf{x}}(t) - \eta\|_Q \right] \frac{1}{2} \right] \left[ \|\tilde{u}_i(t)\|_R^2 \right] dt \]
(A.7)

By (10)–(12), (A.6) and Itô’s formula,
\[ 0 = \mathbb{E} \int_{0}^{T} \sum_{i=1}^{N} e^{-\rho t} \left[ \tilde{\mathbf{x}}_i(t) - \Gamma\tilde{\mathbf{x}}(t) - \eta \right] \frac{1}{2} \left[ \|\tilde{u}_i(t)\|_R^2 \right] dt \]
\[ + N \sum_{i=1}^{N} \left[ \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \tilde{\mathbf{x}}(n) - \tilde{\mathbf{x}}(t) \right) \frac{T}{2} \right] \left[ \|\tilde{\mathbf{x}}_i(t) - \Gamma\tilde{\mathbf{x}}(t) - \eta\|_Q \right] \frac{1}{2} \left[ \|\tilde{u}_i(t)\|_R^2 \right] dt \]

From this and (A.7), we obtain
\[ \left( \begin{array}{c} 1 \int_{0}^{T} e^{-\rho t} \left( \tilde{\mathbf{x}}(n) - \tilde{\mathbf{x}}(t) \right) dt \\ C \Gamma P + PC\tilde{\mathbf{x}}(n) \end{array} \right) dt. \]

By Lemma A.1, (A.4) and (A.5), we obtain
\[ \frac{1}{N} \sum_{i=1}^{N} \tilde{I}_i \leq C E \int_{0}^{T} e^{-\rho t} \left( \|\tilde{\mathbf{x}}(n) - \tilde{\mathbf{x}}(t)\|^2 dt \right) \]
\[ \times \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\tilde{\mathbf{x}}(n) - \tilde{\mathbf{x}}(t)\|^2 dt \right) \]
which implies \( \frac{1}{N} \sum_{i=1}^{N} \tilde{I}_i = O\left(\frac{1}{\sqrt{N}}\right) \).

Moreover, by (10), (13) and direct calculations,
\[ J^{F}_{soc}(\hat{u}) = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\tilde{\mathbf{x}}_i - \Gamma\tilde{\mathbf{x}}(t) - \eta\|_Q^2 + \|\hat{u}_i\|_R^2 \right) dt \]
\[ = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}(n)\|_Q^2 + \|\tilde{\mathbf{x}}_i\|_Q^2 + \|\tilde{\mathbf{x}}(n)\|_Q^2 + \|\eta\|_Q^2 \right. \]
\[ - 2\eta^T Q(I - \Gamma)\tilde{\mathbf{x}}_i + \|\hat{u}_i\|_R^2 \right) dt \]
\[ = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}(n)\|_Q^2 + \|\tilde{\mathbf{x}}_i\|_Q^2 + \|\tilde{\mathbf{x}}(n)\|_Q^2 + \|\eta\|_Q^2 \right) \right. \]
\[ - 2\eta^T Q(I - \Gamma)\tilde{\mathbf{x}}_i + \|\hat{u}_i\|_R^2 \right) dt \]
\[ = \sum_{i=1}^{N} \mathbb{E} \left[ \left. \left( \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}(n)\|_Q^2 + \|\tilde{\mathbf{x}}_i\|_Q^2 + \|\tilde{\mathbf{x}}(n)\|_Q^2 + \|\eta\|_Q^2 \right) - 2\eta^T Q(I - \Gamma)\tilde{\mathbf{x}}_i \right. \right. \]
\[ + \|\hat{u}_i\|_R^2 \right) dt + q_T \]
\[ = \sum_{i=1}^{N} \mathbb{E} \left[ \left. \left( \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}(n)\|_Q^2 + \|\tilde{\mathbf{x}}(n)\|_Q^2 \right) - 2\eta^T Q(I - \Gamma)\tilde{\mathbf{x}}_i \right. \right. \]
\[ + \|\hat{u}_i\|_R^2 \right) dt + q_T \]
\[ = \mathbb{E} \left[ \left. \left( \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}(n)\|_Q^2 + \|\tilde{\mathbf{x}}(n)\|_Q^2 \right) - 2\eta^T Q(I - \Gamma)\tilde{\mathbf{x}}_i \right. \right. \]
\[ + \|\hat{u}_i\|_R^2 \right) dt + q_T \]
which gives that \( J^{F}_{soc}(\hat{u}) = O\left(\frac{1}{\sqrt{N}}\right) \).
Note that $\bar{A} - \frac{e}{T}I$ is Hurwitz. By Schwarz's inequality,
\[
\begin{align*}
\mathbb{E} & \int_0^\infty e^{-\rho t} \|\ddot{x}(t)\|^2 dt \\
& \leq 3\mathbb{E} \int_0^\infty e^{-\rho t} \left( \int_0^t e^{\rho\tau} g(t) e^{\rho\tau} dt \right)^2 dt \\
& + 3\mathbb{E} \int_0^\infty e^{-\rho t} \left( \int_0^t \|e^{\rho\tau} \| \ddot{x}(t)\|^2 dt \right) dt \\
& + 3\mathbb{E} \int_0^\infty e^{-\rho t} \int_0^t [e^{\rho\tau} + e^{\rho\tau}] e^{\rho\tau} \|\ddot{x}(t)\|^2 dt dt \\
& \leq C + 3\mathbb{E} \int_0^\infty e^{-\rho t} \|g(t)\|^2 \int_0^\infty t e^{\rho\tau} \|\ddot{x}(t)\|^2 dt dt \\
& + 3\mathbb{E} \int_0^\infty e^{-\rho t} \|\ddot{x}(t)\|^2 \int_0^\infty e^{\rho\tau} \|\ddot{x}(t)\|^2 dt dt \\
& \leq C.
\end{align*}
\]
This with (20) completes the proof. □

Appendix C. Proofs of Theorems 4.2 and 4.3

Proof of Theorem 4.2. (i)⇒(ii). By (16),
\[
\frac{d\mathbb{E}[\ddot{x}]}{dt} = A\mathbb{E}[\ddot{x}(t)] - BR^{-1}B^T \left( (\Pi T) \ddot{y}(t) + s(t) \right) + GE\mathbb{E}[\dddot{x}(t)] + f(t), \quad \mathbb{E}[\dddot{x}(0)] = \dddot{x}_0.
\]
It follows from (A1) that $\mathbb{E}[\dddot{x}(t)] = \mathbb{E}[\dddot{x}(t)] = \mathbb{E}[\dddot{x}(t)]$. If $a = \frac{T}{2}$, then by comparing (24) and (C.1),
\[
\|\ddot{x}(t)\|^2 = \mathbb{E}[\ddot{x}(t)] \leq \mathbb{E}[\dddot{x}(t)].
\]
By integrating over $[0, T]$ we have
\[
\int_0^T e^{-\rho t} \|\ddot{x}(t)\|^2 dt < \infty.
\]
By (24), we have
\[
\ddot{x}(t) = e^{A + G - BR^{-1}B^T \Pi T} \left[ \dddot{x}_0 + \int_0^t e^{-A + G - BR^{-1}B^T \Pi T} h(\tau) d\tau \right],
\]
where $h = -BR^{-1}B^T + f$. By the arbitrariness of $x_0$, we have
\[
\|\ddot{x}(t)\|^2 \leq \sup_{0 \leq t \leq T} \|x(t)\|.
\]
Then from (27) we have
\[
\mathbb{E} \int_0^\infty e^{-\rho t} \|\dddot{x}(t)\|^2 dt < \infty.
\]
This leads to $\mathbb{E} \int_0^\infty e^{-\rho t} \|g(t)\|^2 dt < \infty$, where $g = -BR^{-1}B^T \left( (\Pi T) + \mathbb{E}[\dddot{x}(t)] + f(t) \right)$. By (B.1), we obtain
\[
\mathbb{E}[\ddot{x}(t)]^2 = \mathbb{E} \left[ e^{A t} \left( x_0 + \int_0^t e^{-A \tau} g(\tau) d\tau \right) \right]^2 \\
+ \mathbb{E} \left[ \int_0^t \left( \|e^{A \tau} + e^{A(\rho\tau - \tau)}\| \|\ddot{x}(t)\|^2 \right) \right] dt.
\]
By (27) and the arbitrariness of $\dddot{x}_0$, we obtain that $\bar{A} - \frac{e}{T}I$ is Hurwitz, i.e., $\bar{A} - \frac{e}{T}I$ is Hurwitz. By Anderson and Moore's (1990), (21) and (22) admit a unique solution such that $P > 0$. From (C.2) and (C.3),
\[
\mathbb{E} \int_0^\infty e^{-\rho t} \|\dddot{x}(t)\|^2 dt < \infty.
\]
On the other hand, (A.2) gives
\[
\begin{align*}
\mathbb{E} \|\dddot{x}(t)\|^2 &= \mathbb{E} \left[ \left( e^{A t} \right) \left( \dddot{x}(t) \right) \right]^2 \\
& + \frac{1}{2} \left( \int_0^t \mathbb{E} \left[ \|e^{A \tau} + e^{A(\rho\tau - \tau)}\| \|\ddot{x}(t)\|^2 \right] \right) dt.
\end{align*}
\]
By (C.4) and the arbitrariness of $x_0$, we obtain that $\bar{A} + G - \frac{e}{T}I$ is Hurwitz.

Denote $V$ by $V^*$ when $\dddot{x} = -BR^{-1}B^T \Pi T$. By (22),
\[
\begin{align*}
\frac{dV^*}{dt} &= \dddot{x}(t)^T \left( -\rho \Pi + A + G - BR^{-1}B^T \Pi T \right) \Pi T \\
& + \Pi \left( A + G - BR^{-1}B^T \Pi T \right) \dddot{x}(t) \Pi T \\
& = -\rho t (Q + \mathbb{E}[\dddot{x}(t)]^2 \Pi T \leq 0.
\end{align*}
\]
Note that $V^* \geq 0$. Then $\lim_{t \to \infty} V^*(t)$ exists, which implies
\[
\lim_{t \to \infty} V^*(t) = V^*(t_0 + T).
\]
Rewrite $\Pi T$ in (13) by $\Pi T$. Then we have $\Pi T_{t_0}$.

This with (C.5) implies
\[
\lim_{t_0 \to \infty} \frac{d}{dt} \left( \Pi T_{t_0} \right) = \Pi T_{t_0}(0).
\]
By (A3), one can get that there exists $T > 0$ such that $\Pi T_{t_0} > 0$ (see e.g. Zhang et al. (2019) and Zhang, Zhang, and Chen (2008)). Thus, we have $\lim_{t_0 \to \infty} e^{-\rho t_0} \|\dddot{x}(t_0)\|^2 = 0$, which implies that $\bar{A} + G - \frac{e}{T}I$ is stabilizable. Similarly, we can show that $\bar{A} - \frac{e}{T}I$ is stabilizable.

Proof of Theorem 4.3. (ii)⇒(i). From Anderson and Moore (1990), (21) and (22) admit a unique solution such that $0 \geq B - BR^{-1}B^T + \frac{e}{T}I$ are Hurwitz, respectively. Thus, there exists a unique solution $x(t)$ such that $x(t) \in C_{\mathbb{R}}(0, \infty, \mathbb{R}^n)$. It is straightforward that $\bar{x} \in C_{\mathbb{R}}(0, \infty, \mathbb{R}^n)$. By the argument in the proof of Theorem 4.1, (i) follows. (ii)⇒(iii). The proof of this part is similar to that of (i)⇒(ii) in Theorem 4.2.

(iii)⇒(i). Since $\Pi T > 0$, there exists an orthogonal $U$ such that $U^T \Pi U = \left[ \begin{array}{cc} 0 & 0 \\ 0 & \Pi T \end{array} \right]$, where $\Pi T > 0$. From (21),
\[
\rho U^T \Pi U = (U^T \bar{A} \bar{U})^T U^T \Pi U + U^T \Pi UU^T \bar{A} \bar{U} + U^T \bar{Q} \bar{U},
\]
where $\bar{A} \bar{U} = A + G - BR^{-1}B^T \Pi T, \bar{Q} = Q + \bar{Q} + \Pi BR^{-1}B^T \Pi T$. Denote
\[
U^T \bar{A} \bar{U} = \left[ \begin{array}{ccc} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{array} \right], \quad U^T \bar{Q} \bar{U} = \left[ \begin{array}{ccc} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{array} \right].
\]
By pre- and post-multiplying by $\xi^T$ and $\xi$ where $\xi = [\xi_1, 0]^T$, it follows that

$$0 = \rho \xi^T \Pi U \xi = \xi^T \Pi U \xi.$$  

From the arbitrariness of $\xi_1$, we obtain $Q_{11} = 0$. Since $Q$ is semi-positive definite, then $Q_{12} = Q_{21} = 0$, and $Q_{22} \geq 0$. By comparing each block matrix of both sides of (C.6), we obtain $A_{21} = 0$. It follows from (C.6) that $\rho \Pi_2 = \Pi_1 \tilde{A}_{22} + \tilde{A}_{21}^T \Pi_1 + \tilde{Q}_{22}$. Let $\xi = [\xi_1^T, \xi_2^T]^T = U^T \tilde{y}$, where $\tilde{y}$ satisfies $\dot{\tilde{y}} = \tilde{A}\tilde{y}$. Then we have

$$\dot{\xi}_1 = \tilde{A}_{11} \xi_1 + \tilde{A}_{12} \xi_2,$$  

$$\dot{\xi}_2 = \tilde{A}_{22} \xi_2.$$  

By Lemma 4.1 of Wonham [1968], the detectability of $(A + G, [Q - \Sigma]/2)^T$ implies the detectability of $(\tilde{A}, \tilde{Q}_{11}^T/2)$. Take $\zeta(0) = \xi = [\xi_1, 0]^T$. Then $\tilde{Q}_{11}^{1/2} \tilde{y} = \tilde{Q}_{11}^{1/2} U \xi = 0$, which together with the detectability of $(\tilde{A}, \tilde{Q}_{11}^T/2)$ implies $\zeta_1 \to 0$ and $\tilde{A}_{11}$ is Hurwitz. Denote $S(t) = e^{-\tilde{A}_1 t} \tilde{Q}_{22} e^{\tilde{A}_1 t}$. By (C.7),

$$S(T) - S(0) = -\int_0^T \tau(\tilde{Q}_{22} \tilde{z}(t) - \tilde{z}(t) \tilde{Q}_{22} \tilde{z}(t)) dt \leq 0,$$

which implies $\lim_{t \to \infty} S(t)$ exists. By a similar argument with the proof of Theorem 4.2, we obtain $\lim_{t \to \infty} e^{-\rho t} \|\zeta_2(t)\|_2^2$ and $\Pi_{2,T}(0)$ are both greater than 0, which gives $\tilde{z}_2 \to 0$ and $\tilde{A}_{22}$ is Hurwitz. This with the fact that $A_{11}$ is Hurwitz gives that $\xi_1$ is stable, which leads to (iii). □

References

Bing-Chang Wang received the Ph.D. degree in System Theory from Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2011. From September 2011 to August 2012, he was with Department of Electrical and Computer Engineering, University of Alberta, Canada, as a Postdoctoral Fellow. From September 2012 to September 2013, he was with School of Electrical Engineering and Computer Science, University of Newcastle, Australia, as a Research Academic. From October 2013, he has been with School of Control Science and Engineering, Shandong University, China, as an associate Professor. He held visiting appointments as a Research Associate with Carleton University, Canada, from November 2014 to May 2015, and with the Hong Kong Polytechnic University from November 2016 to January 2017. His current research interests include mean field games, stochastic control, multiagent systems and event based control. He received the IEEE CSS Beijing Chapter Young Author Prize in 2018.

Huanshui Zhang received the B.S. degree in mathematics from Qufu Normal University, Shandong, China, in 1986, the M.Sc. degree in control theory from Heilongjiang University, Harbin, China, in 1991, and the Ph.D. degree in control theory from Northeastern University, Shenyang, China, in 1997. He was a Postdoctoral Fellow at Nanyang Technological University, Singapore, from 1998 to 2001 and Research Fellow at Hong Kong Polytechnic University, Hong Kong, China, from 2001 to 2003. He currently holds a Professorship at Shandong University of Science and Technology, Qingdao, China. He was a Professor with the Harbin Institute of Technology, Shenzhen, China, from 2003 to 2006 and a Professor with Shandong University, Jinan, China, from 2006 to 2019. He also held visiting appointments as a Research Scientist and Fellow with Nanyang Technological University, Curtin University of Technology, and Hong Kong City University from 2003 to 2006. His interests include optimal estimation and control, time-delay systems, stochastic systems, signal processing and wireless sensor networked systems.

Ji-Feng Zhang received the B.S. degree in mathematics from Shandong University, China, in 1985 and the Ph.D. degree from the Institute of Systems Science (ISS), Chinese Academy of Sciences (CAS), China, in 1991. Since 1985, he has been with the ISS, CAS, and now is the Director of ISS. His current research interests include system modeling, adaptive control, stochastic systems, and multi-agent systems.

Dr. Zhang is an IEEE Fellow, IFAC Fellow, CAA Fellow, member of the European Academy of Sciences and Arts, and Academician of the International Academy for Systems and Cybernetic Sciences. He received twice the Second Prize of the State Natural Science Award of China in 2010 and 2015, respectively. He is a Vice-President of the Chinese Association of Automation, Vice-President of the Chinese Mathematical Society, Associate Editor-in-Chief of Science China Information Sciences, and Senior Editor of IEEE Control Systems Letters. He was a Vice-Chair of the IFAC Technical Board, member of the Board of Governors, IEEE Control Systems Society; Convenor of Systems Science Discipline, Academic Degree Committee of the State Council of China; Vice-President of the Systems Engineering Society of China; and Editor-in-Chief, Deputy Editor-in-Chief or Associate Editor of more than 10 journals, including Journal of Systems Science and Mathematical Sciences, IEEE Transactions on Automatic Control and SIAM Journal on Control and Optimization etc. He was General Co-Chair or IPC Chair of many control conferences.